Gold And Crude Oil As Safe-haven Assets For Clean Energy Stock Indices: Blended Copulas Approach

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Commodity Volatility Shocks And BRIC Sovereign Risk: A GARCH-Quantile Approach

Gold and crude oil as safe-haven assets for clean energy stock indices: Blended copulas approach

Bouri Elie a, Jalkh Naji b, Anupam Dutta c, Gazi Salah Uddin d, e
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\[ R_{k,t} = \mu_{k,t} + \varepsilon_{k,t} \]

\[ \mu_{k,t} = \mu_k + \sum_{i=0}^{P_{1,k}} \Phi_{k,i} (R_{k,t-i} - \mu_k) + \sum_{j=0}^{q_{1,k}} \theta_{k,j} \varepsilon_{k,t-j} \]

\[ \varepsilon_{k,t} = \sigma_{k,t} Z_{k,t} \]

\[ Z_{k,t} \sim \mathcal{N}(0,1) \]

\[ \sigma_{k,t}^2 = \omega_k + \sum_{i=0}^{P_{2,k}} \alpha_{k,i} (|\varepsilon_{k,t-i}| - \gamma_{k,i} \varepsilon_{k,t-i})^2 + \sum_{j=0}^{q_{2,k}} \beta_{k,j} \sigma_{k,t-j}^2, -1 < \gamma_{k,i} < 1 \]
Gold And Crude Oil As Safe-haven Assets For Clean Energy Stock Indices: Blended Copulas Approach

Motivation Behind The Study and Literature Review

Data Presentation and Discussion

The Research Methodology

Empirical Results
Motivation Behind The Study and Literature Review

Gold And Crude Oil As Safe-haven Assets For Clean Energy Stock Indices: Blended Copulas Approach
We examine the potential roles of two strategic commodities, GOLD and CRUDE OIL, as safe-haven assets against extreme down movements in two Clean Energy Stock Indices, the S&P Global Clean Energy and the WilderHill Clean Energy.

Instead of adopting a priori selection of the best copula function, we propose a flexible and rich tail dependency modelling approach, i.e. mixture of copulas to better illustrate the dependence between the pairs of variables under study.

We also apply parametric as well as non-parametric tail dependencies measures.
Motivation Behind The Study and Literature Review

Although no theoretical models can explain the relationship between Clean Energy Stock Indices and Gold Or Crude Oil,

<table>
<thead>
<tr>
<th>Different econometric methods have been used such as DCC and Multivariate GARCH models (Ahmad et al., 2018; Sadorsky, 2012a), Connectedness measures (Ahmad, 2017), and Cointegration (Bondia et al., 2016).</th>
</tr>
</thead>
</table>

Generally, these methods fail to account for the tail dependence which is crucial for understanding the extreme market co-movements and thus for making appropriate inferences on the potential role of gold and crude oil as safe-haven assets when such a role is needed the most by investors.
Motivation Behind The Study and Literature Review

- The literature testing safe-haven role of Gold and Crude Oil for Clean Energy Stock Indices is very limited.

Strands Of Research

- A first strand of research examines the relation amongst crude oil and (clean) energy stock indices.
- Another strand of literature examines the association between gold and stock markets.
Motivation Behind The Study and Literature Review - A First Strand Of Research

<table>
<thead>
<tr>
<th>Authors</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Henriques and Sadorsky (2008)</td>
<td>are the first to <strong>assess</strong> the connection between crude oil and <strong>renewable energy equity markets</strong>. Employing a vector VAR model, the authors find that clean energy stock returns are affected by variations in oil prices.</td>
</tr>
<tr>
<td>Sadorsky (2012a)</td>
<td>uses a set of multivariate GARCH models to examine if <strong>global crude oil market sends volatility</strong> to <strong>clean energy firm returns</strong>. This study also concludes that crude oil and alternative energy markets co-move.</td>
</tr>
<tr>
<td>Kumar et al. (2012)</td>
<td>study whether an increase in fossil fuel price would cause a significant growth in clean energy investments. Using several new energy equity indices, the authors find that fuel prices tend to positively impact the returns of clean energy stocks.</td>
</tr>
</tbody>
</table>
Motivation Behind The Study and Literature Review - A First Strand Of Research

| Broadstock et al. (2012) | consider the application of time-varying correlation to assess whether fossil fuel prices influence the prices of energy related stocks in China. The authors find a much stronger association following the US subprime crisis of 2007-2008. This significant linkage suggests that the Chinese equity markets, particularly the energy related stocks, are highly sensitive to global oil price shocks. |
| Sadorsky (2012b) | also shows that an upsurge in international oil prices tends to decrease the risks of clean energy companies. |
| Managi and Okimoto (2013) | utilize the Markov-Switching VAR models and provide evidence of a structural change in late 2007 in the association across the prices of crude oil, clean energy stocks and technology stocks. The authors also find that during the 2007 break period there was a significant increase in the price of oil, and that world oil prices and clean energy stock prices move in the same direction after the structural break. |
Motivation Behind The Study and Literature Review - A First Strand Of Research

Bondia et al. (2016) explore the long-run association between the returns of global oil prices and clean energy stock indices using a multivariate framework. At the empirical stage, the authors employ **threshold co-integration tests** and confirm the existence of long-term linkage with two endogenous structural breaks.

Reboredo et al. (2017) investigate whether the international oil prices and new energy stock prices are correlated. Applying **continuous and discrete wavelets**, the study finds weak linkage between the variables in the short run. However, such relationship becomes robust in the long run, mainly for the post global financial era.

A more recent investigation by Dutta (2017) shows that oil price volatility, an indicator of oil price uncertainty, has emerged as a significant factor in predicting the realized volatility of clean energy stocks.
## Motivation Behind The Study and Literature Review – The Second Strand Of Research

<table>
<thead>
<tr>
<th>Authors</th>
<th>Summary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baur and Lucey (2010)</td>
<td>investigate the association between gold returns and US, UK and German stock and bond returns to <em>study the hedge and safe-haven abilities of the yellow metal</em>. The authors document that gold is a hedge for stocks and a safe-haven in stress periods.</td>
</tr>
<tr>
<td>Baur and McDermott (2010)</td>
<td>show that gold is a safe-haven for only major European stock markets and the US.</td>
</tr>
<tr>
<td>Mensi et al. (2015)</td>
<td>find strong evidence of diversification benefits and hedging effectiveness when buying assets in gold and stock markets.</td>
</tr>
<tr>
<td>Arouri et al. (2015)</td>
<td>note that combining gold and Chinese stocks leads to a better risk-adjusted return. Further, during the crisis periods, gold has acted as a safe-haven asset.</td>
</tr>
</tbody>
</table>
Motivation Behind The Study and Literature Review

In addition to comparing between the safe-haven abilities of both Gold & Crude Oil against two Clean Energy Stock Indices (S&P Global Clean Energy and the WilderHill Clean Energy)

Our study differs from the above-mentioned studies in its reliance on Single & Mixture Copulas to describe the Dependence Structure.

Importantly, it applies parametric as well as non-parametric tail dependencies measures.

This combination of Tail Dependency Measures and Mixture Copulas nicely extends to the related literature on the role of those two strategic commodities as safe-haven assets against Clean Energy Stock Indices.
\[ R_{k,t} = \mu_{k,t} + \varepsilon_{k,t} \]

\[ \mu_{k,t} = \mu_k + \sum_{i=0}^{P_{1,k}} \phi_{k,i} (R_{k,t-i} - \mu_k) + \sum_{j=0}^{q_{1,k}} \theta_{k,j} \varepsilon_{k,t-j} \]

\[ \varepsilon_{k,t} = \sigma_{k,t} Z_{k,t} \]

\[ Z_{k,t} \sim \mathcal{N}(0,1) \]

\[ \sigma^2_{k,t} = \omega_k + \sum_{i=0}^{P_{2,k}} \alpha_{k,i} \left( |\varepsilon_{k,t-i} - \gamma_{k,i} \varepsilon_{k,t-i}| \right)^2 + \sum_{j=0}^{q_{2,k}} \beta_{k,j} \sigma^2_{k,t-j}, -1 < \gamma_{k,i} < 1 \]
This study is conducted with daily data. The analysis involves two clean energy indices: the S&P Global Clean Energy Index (SPGTCED) and the WilderHill Clean Energy Index (ECO). Additionally, the spot prices of crude oil represented by the West Texas Intermediate (WTI) and the spot prices of one ounce of Gold (GOLD) are considered.

All data were extracted from DataStream (3746 daily observations). Due to data availability, the entire dataset covers the period from November 21st, 2003 to March 30th, 2018.
Data Presentation and Discussion

**Market Data At A Glance**

**S&P Global Tradable Clean Energy Index (SPGTCED)**

- By S&P Dow Jones,
- Liquid and tradable exposure to a diversified blend of 30 companies involved in clean energy-related businesses, i.e. utilities (51.3%), industrials (25.5%), information technologies (20.8%), and materials (2.4%).

**ECO Index**

- Modified equal-dollar weighting Index
- Tracks the Clean Energy sector, via firms identified to have substantial exposure to, or contributing to the development of clean energy in an ecological and economic sensible ways,
- Companies that would benefit *significantly from a communal transition* towards a cleaner energy lifestyle.
- Encompasses companies in energy-related businesses, such as: Renewable Energy Supplies, Power & Energy Delivery and/or Storage, Clean Fuels, as well as Greener Utilities
Data Presentation and Discussion

Market Data At A Glance

- It appears that, except for GOLD, all series experienced a sharp decline during the global financial crisis (GFC) of 2008.

- Importantly, in the post GFC period, WTI rebounded abruptly but failed to attain its pre-GFC top; whereas GOLD succeeded in making higher highs before peaking in 2012.
Data Presentation and Discussion

Data At A Glance

- WTI is riskier than ECO, SPGTCED and GOLD.

- Although SPGTCED is relatively less volatile than WTI and ECO, it exhibits an extremely high kurtosis and a lower minimum, revealing the existence and/or the propensity to produce extreme negative returns.

- Except for WTI, all return series suffer from negative skewness, implying asymmetric and fat-tail behavior.

- Ljung–Box Q and Q² tests reveal that the unconditional distributions of asset returns depart from normality, suffer from serial correlation, and exhibit ARCH effects.

### Table 1: Descriptive, autocorrelation and ARCH statistics of daily returns

<table>
<thead>
<tr>
<th></th>
<th>SPGTCED</th>
<th>ECO</th>
<th>WTI</th>
<th>GOLD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Descriptive statistics</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean (%)</td>
<td>-0.013368</td>
<td>-0.025740</td>
<td>0.018653</td>
<td>0.032215</td>
</tr>
<tr>
<td>Maximum (%)</td>
<td>18.09272</td>
<td>14.51950</td>
<td>16.41370</td>
<td>8.590126</td>
</tr>
<tr>
<td>Std. Dev. (%)</td>
<td>1.817574</td>
<td>1.972403</td>
<td>2.326364</td>
<td>1.154994</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.549906</td>
<td>-0.374195</td>
<td>0.043260</td>
<td>-0.376803</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>17.07976</td>
<td>8.534472</td>
<td>7.616092</td>
<td>8.568422</td>
</tr>
<tr>
<td>Panel B: Autocorrelation and ARCH statistics</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>31122.39</td>
<td>4867.012</td>
<td>3326.152</td>
<td>4927.054</td>
</tr>
<tr>
<td>Probability</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>Ljung-Box Q test of serial correlation (p-value)</td>
<td>0.0002369</td>
<td>0.0007448</td>
<td>2.972e-06</td>
<td>0.004428</td>
</tr>
<tr>
<td>Ljung-Box Q² test of ARCH Effect (p-value)</td>
<td>&lt; 2.2e-16</td>
<td>&lt; 2.2e-16</td>
<td>&lt; 2.2e-16</td>
<td>&lt; 2.2e-16</td>
</tr>
<tr>
<td>Observations</td>
<td>3745</td>
<td>3745</td>
<td>3745</td>
<td>3745</td>
</tr>
</tbody>
</table>
Data Presentation and Discussion

Data At A Glance

Most of the distributions do not seem to have *elliptical* structural shape.

Need to jointly handle the *dependence* & the *marginal* distributions by means of *Copula*. 
The Research Methodology

Gold And Crude Oil As Safe-haven Assets For Clean Energy Stock Indices: Blended Copulas Approach

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\[ \sigma_{k,t}^2 = \omega_k + \sum_{i=0}^{p_{2,k}} \alpha_{k,i}(|\varepsilon_{k,t-i} - \gamma_{k,i}\varepsilon_{k,t-i}|)^2 + \sum_{j=0}^{q_{2,k}} \beta_{k,j}\sigma_{k,t-j}^2, -1 < \gamma_{k,i} < 1 \]
The Research Methodology

A Two-step Approach: Following Ji et al., 2018

The First Stage

• Best fitting of mean and variance equations for standard and GJR-ARMA-GARCH to all series.

• Filtering standardized residuals from the ARMA-GJR-GARCH models

The Second Stage

• Apply single and mixture copulas to efficiently study the dependence between gold (crude oil) and clean energy stock indices

The Research Methodology

The First Stage

• We use the asymmetric GARCH of Glosten et al. (1993), a.k.a. GJR-GARCH model, which adds an asymmetric term to capture the asymmetric response of the conditional variance to shocks.

• To decide which GARCH-based model has a superior fit and specifications, i.e. symmetric (GARCH) or asymmetric (GJR-GARCH); and the density of the error distribution, normal, skewed normal, t-student, skewed t-student, generalized error distributions or skewed GED, we follow Ji et al., 2018 and use SIC.
The mean and variance equations of ARMA-GJR-GARCH models have the below form:

\[ R_{k,t} = \mu_{k,t} + \varepsilon_{k,t} \]  
\[ \mu_{k,t} = \mu_k + \sum_{i=0}^{p_{1,k}} \phi_{ki}(R_{k,t-i} - \mu_k) + \sum_{j=0}^{q_{1,k}} \theta_{kj}\varepsilon_{k,t-j} \]  
\[ \varepsilon_{k,t} = \sigma_{k,t}Z_{k,t} \]  
\[ Z_{k,t} \sim \mathcal{N}_{k,v}(0,1) \]  
\[ \sigma_{k,t}^2 = \omega_k + \sum_{i=0}^{p_{2,k}} \alpha_{ki}(|\varepsilon_{k,t-i}| - \gamma_{ki}\varepsilon_{k,t-i})^2 + \sum_{j=0}^{q_{2,k}} \beta_{kj}\sigma_{k,t-j}^2, \quad -1 < \gamma_{ki} < 1 \]

where \( Z_{k,t}, k \in \mathbb{N}, t \in \mathbb{Z} \) denotes a random variable that follows a \( \mathcal{N}_{k,v} \)-distribution with a mean value of zero and variance of one, and where the return of the \( k^{th} \) asset contained in the portfolio, \( k = 1, \ldots, K \) is denoted by \( R_{k,t} \).

In Equation (5), \( \gamma_{k,i} \) represents the asymmetric term.
The Research Methodology

A Two-step Approach: Following Ji et al., 2018

The First Stage
• Best fitting of mean and variance equations for standard and GJR-ARMA-GARCH to all series.
• Filtering standardized residuals from the ARMA-GJR-GARCH models

The Second Stage
• Apply single and mixture copulas to efficiently study the dependence between gold (crude oil) and clean energy stock indices

The Research Methodology (Cont.)

The Second Stage
• A semi-parametric Inference Functions for Margins (IFM) approach is implemented for selecting the appropriate bivariate copula family.
• Copula family will be selected according to the BIC.
The Research Methodology

The Second Stage: A Closer Look

- We extend the ARMA-GJR-GARCH–Copula approach by incorporating mixture of copulas. For that purpose, a mixture of m d-dimensional copulas with weights \( w_j, j = 1, 2, ..., m \) is itself a d-dimensional copula, having a joint CDF and PDF:

\[
C(X; \xi) = \sum_{j=1}^{m} w_j \times C_j(X, \Theta_j)
\]

\[
c(X; \xi) = \sum_{j=1}^{m} w_j \times c_j(X, \Theta_j)
\]

- where \( C_j \) and \( c_j \) denote respectively the CDFs and the densities of the m component copulas, \( j=1, 2, ..., m \), and \( \xi = (W, \Theta) \)

- Best copula mixtures are selected using a maximum log-likelihood composite function, which is given by:

\[
\mathcal{L}(\xi, c(\hat{Z}))
\]
The Research Methodology – 40 Copula Families & 1520 of Mixture Copulas Combinations.

<table>
<thead>
<tr>
<th>0 = Independence copula</th>
<th>8 = BB6 copula</th>
<th>19 = rotated BB7 copula (180 degrees; “survival BB7”)</th>
<th>30 = rotated BB8 copula (90 degrees)</th>
<th>104 = Tawn type 1 copula</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 = Gaussian copula</td>
<td>9 = BB7 copula</td>
<td>20 = rotated BB8 copula (180 degrees; &quot;survival BB8&quot;)</td>
<td>33 = rotated Clayton copula (270 degrees)</td>
<td>114 = rotated Tawn type 1 copula (180 degrees)</td>
</tr>
<tr>
<td>2 = Student t copula (t-copula)</td>
<td>10 = BB8 copula</td>
<td>23 = rotated Clayton copula (90 degrees)</td>
<td>34 = rotated Gumbel copula (270 degrees)</td>
<td>124 = rotated Tawn type 1 copula (90 degrees)</td>
</tr>
<tr>
<td>3 = Clayton copula</td>
<td>13 = rotated Clayton copula (180 degrees; “survival Clayton”)</td>
<td>24 = rotated Gumbel copula (90 degrees)</td>
<td>36 = rotated Joe copula (270 degrees)</td>
<td>134 = rotated Tawn type 1 copula (270 degrees)</td>
</tr>
<tr>
<td>4 = Gumbel copula</td>
<td>14 = rotated Gumbel copula (180 degrees; &quot;survival Gumbel&quot;)</td>
<td>26 = rotated Joe copula (90 degrees)</td>
<td>37 = rotated BB1 copula (270 degrees)</td>
<td>204 = Tawn type 2 copula</td>
</tr>
<tr>
<td>5 = Frank copula</td>
<td>16 = rotated Joe copula (180 degrees; “survival Joe”)</td>
<td>27 = rotated BB1 copula (90 degrees)</td>
<td>38 = rotated BB6 copula (270 degrees)</td>
<td>214 = rotated Tawn type 2 copula (180 degrees)</td>
</tr>
<tr>
<td>6 = Joe copula</td>
<td>17 = rotated BB1 copula (180 degrees; “survival BB1”)</td>
<td>28 = rotated BB6 copula (90 degrees)</td>
<td>39 = rotated BB7 copula (270 degrees)</td>
<td>224 = rotated Tawn type 2 copula (90 degrees)</td>
</tr>
<tr>
<td>7 = BB1 copula</td>
<td>18 = rotated BB6 copula (180 degrees; &quot;survival BB6&quot;)</td>
<td>29 = rotated BB7 copula (90 degrees)</td>
<td>40 = rotated BB8 copula (270 degrees)</td>
<td>234 = rotated Tawn type 2 copula (270 degrees)</td>
</tr>
</tbody>
</table>
The Research Methodology

Asymptotic Tail Dependence Coefficients (ATDC): A Parametric Approach

- The Asymptotic upper tail dependence coefficient denoted by $\lambda_U(\tau)$ (upper ATDC) is defined as the limit, when it exists, of the conditional probability of $Y$, where the latter is greater than the 100$\tau$-th percentile of $F_Y$ given that $X$ is greater than the 100$\tau$-th percentile of $F_X$ as $\tau$ tends to 1, i.e.

  $$\lambda_U(\tau) = \lim_{\tau \to 1^-} Pr\left\{ Y > F_Y^{(-1)}(\tau) | X > F_X^{(-1)}(\tau) \right\}$$

- The Asymptotic lower tail dependence coefficient denoted by $\lambda_L(\tau)$ (Lower ATDC) is defined as the limit, when it exists, of the conditional probability of $Y$, where the latter is less than or equal to the 100$\tau$-th percentile of $F_Y$ given that $X$ is less than or equal to the 100$\tau$-th percentile of $F_X$ as $\tau$ tends to 0, i.e.

  $$\lambda_L(\tau) = \lim_{\tau \to 0^+} Pr\left\{ Y \leq F_Y^{(-1)}(\tau) | X \leq F_X^{(-1)}(\tau) \right\}$$
Extreme Tail Dependence Coefficients: A Non-Parametric Approach

However, as clearly indicated in Bouyé (2009), independence in the sense of “factorization of a bivariate distribution in the tails” would lead $\lambda_U$ to be equal to zero, but the reverse is not true.

Hence, even in the presence of an asymptotic lower tail dependence coefficient being equal to zero, there may still be tail dependence.

• Usually, a non-parametric based estimator denoted by $\hat{\lambda}_L^2$, is implemented for the purposes of quantifying the strength of dependence in lower and upper tails. $\hat{\lambda}_L^2$ is defined as follows:

\[
\hat{\lambda}_L^2 = \left( \sum_{i=1}^{k} \left( \frac{i}{n} \right)^2 \right)^{-1} \sum_{i=1}^{k} \left( \frac{i}{n} \hat{C} \left( \frac{i}{n} \right) \right)
\]

• We extend the application of the least squares estimator ($\hat{\lambda}_L^2$) to embrace best selected copula function, whether the latter is a single copula family or a mixture of copula functions.

The Research Methodology
Empirical Results

Gold And Crude Oil As Safe-haven Assets For Clean Energy Stock Indices: Blended Copulas Approach
Empirical Results

• Panel A in Table 7 reports the Asymptotic Lower Tail Dependence associated with best copula mixture for every paired assets. It reveals that Gold & WTI are archetypal “safe haven” for S&P Global Clean Energy Index, with WTI being an exceptional candidate for fulfilling that role.

• However, Panel B reports non-parametric-based Extreme Lower Tail Dependence coefficients ($\lambda^2_L$) discussed earlier. Results confirm that Gold & WTI will typically serve as “weak safe haven” for S&P Global Clean Energy Index, and endorse the superiority of WTI. However, when examining the WilderHill Clean Energy Index instead, results validate previous conclusion but with the Gold taking over WTI.
<table>
<thead>
<tr>
<th>Market Participants</th>
<th>Building an optimal portfolio in order to gain superior risk-adjusted returns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Predicting future clean stock return volatility</td>
</tr>
<tr>
<td></td>
<td>Achieving better portfolio diversification benefits</td>
</tr>
<tr>
<td></td>
<td>Improve their hedging performances</td>
</tr>
<tr>
<td>Policy-Makers</td>
<td>Develop effective strategies for reducing the adverse effect of oil price volatility</td>
</tr>
<tr>
<td></td>
<td>Attract environmentally conscious investors and create a better investment condition for firms operating in renewable energy sectors</td>
</tr>
<tr>
<td>Researchers</td>
<td>Develop appropriate asset-pricing models</td>
</tr>
<tr>
<td></td>
<td><strong>Improve</strong> volatility prediction methods to achieve proper knowledge of shock transmission across different financial markets.</td>
</tr>
</tbody>
</table>
\[ R_{k,t} = \mu_{k,t} + \varepsilon_{k,t} \]
\[ \mu_{k,t} = \mu_k + \sum_{i=0}^{p_2,k} \phi_{ki} (R_{k,t-i} - \mu_k) + \sum_{j=0}^{q_1,k} \theta_{kj} \varepsilon_{k,t-j} \]
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\[ \sigma_{k,t}^2 = \omega_k + \sum_{i=0}^{p_2,k} \alpha_{ki} \left( |\varepsilon_{k,t-i}| - \gamma_{ki} \varepsilon_{k,t-i} \right)^2 + \sum_{j=0}^{q_2,k} \beta_{kj} \sigma_{k,t-j}^2, -1 < \gamma_{ki} < 1 \]